TEACHER NOTES: PRODUCT OF DIGITS TRICK

This trick could be used as a hook for a divisibility test lesson.

But it could also be used with older students. A tricky probability problem (see below) arises from the trick, that would be a challenge for more able students at GCSE (or perhaps A-Level students).

Instructions for students:

Multiply 10 randomly chosen digits together.

You can't include 0, and you can (you'll have to) repeat some digits.

Choose one of the digits in the display to be your secret digit. It must be a non-zero digit.

Then read all your digits apart from your secret digit.

The mathemagician will tell you your secret digit.

Instructions for the 'mathemagician':

The number they have made has a 92.5% chance of being a multiple of 9.

The digits of any multiple of 9 add up to a multiple of 9.

Add up the digits the student reads.

Their secret digit is very likely to be whatever you need to add on to their total to make the next multiple of 9. Zero cannot be their secret digit.

WARNING: THIS TRICK IS NOT CERTAIN TO WORK! (Perhaps choose a student who looks like their hand is moving a lot across the calculator pad).

Probability maths

What is the chance that someone (who is choosing their digits randomly) will create a multiple of 9?

This would be a very long tree diagram to actually draw out (there are 10 events), but we could use the same principles of probability without drawing out the tree diagram.

Solution

An easier thing to work out is the chance that a multiple of 9 is not made.

To make a number that is not a multiple of 9, you need to **not pick 9**, of course, but also you need to ensure you get **no more than one factor of 3**.

This means that, generally, there are two ways you can make a number that is not multiple of 9.

Either, you pick 10 digits from the set $\{1,2,4,5,7,8\}$
OR
You pick 9 digits from the set $\{1,2,4,5,7,8\}$ AND only 1 digit from the set $\{3,6\}$

The language of 'one thing OR another means you ADD, one thing AND another means you MULTIPLY', may help students deduce the expression below.

It follows that:

$$P(not\ a\ multiple\ of\ 9) = \left(\frac{6}{9}\right)^{10} + \left[\left(\frac{6}{9}\right)^{9} \times \frac{2}{9} \times 10\right]$$

There is a factor of 10 in the second expression because the '3 or 6' can be any one of the 10 digits.

It may not seem obvious to students where that factor of 10 comes from. One way to explain it would be to refer to tree diagrams.

Imagine a tree diagram with 10 events, where each event is a choice between 'choosing 3 or 6' (C) or 'not choosing 3 or 6' (NC).

If you were to draw out the whole tree diagram there would be 10 routes that result in 'not choosing 3 or 6' (NC) 9 times and 'choosing 3 or 6' (C) once.

Eg
C,NC,NC,NC,NC,NC,NC,NC,NC,NC
Or
NC,C, NC,NC,NC,NC,NC,NC,NC,NC

And so on.

Each of these routes has a chance of: $\left(\frac{6}{9}\right)^{9} \times \frac{2}{9}$

Adding the chance of each of these routes together is the same as:
$\left(\frac{6}{9}\right)^{9} \times \frac{2}{9} \times 10$

Now, we want to find the chance you do make a multiple of 9.

$$P(multiple\ of\ 9) = 1 - \left[\left(\frac{6}{9}\right)^{10} + \left[\left(\frac{6}{9}\right)^{9} \times \frac{2}{9} \times 10\right]\right]$$

This equates to approximately 92.5%.