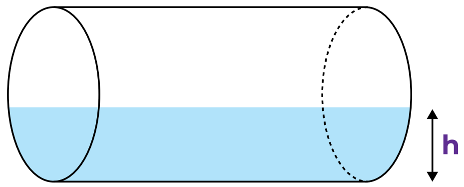
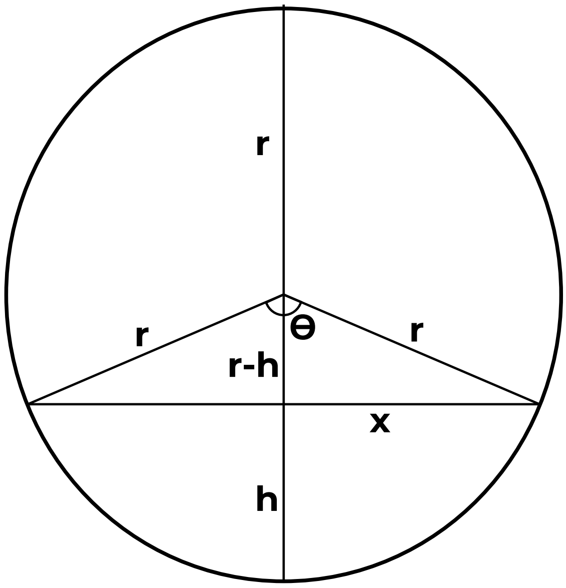
Maths Fest Puzzles - Solutions

Puzzle 1

The diagram to the right shows the cross-section of the cylinder with radius **r**. The bold horizontal line is the fluid level '**h**' above the base of the cylinder.



Our solution rests on finding the area of the segment taken up by the liquid. The solution assumes the angle is in radians, not degrees, but if you want to work in degrees you can use the same method but replace $\frac{\theta r^2}{2}$ with $\frac{\theta r^2}{360}$.



Area of sector: $\frac{\theta r^2}{2}$

Area of the isosceles triangle: $(r-h)x$

Area of segment: $\frac{\theta r^2}{2} - (r-h)x$

The length x can be written in terms of r and h using Pythagoras's theorem: $x = \sqrt{r^2 - (r-h)^2}$

And then simplifies to: $x = \sqrt{h(2r-h)}$

The angle can be found in terms of r and h, by considering the right angle triangle made when the vertical line cuts the isosceles triangle in half:

$$\cos\left(\frac{\theta}{2}\right) = \frac{r-h}{r} \qquad \frac{\theta}{2} = \cos^{-1}\left(\frac{r-h}{r}\right) \qquad \theta = 2\cos^{-1}\left(\frac{r-h}{r}\right)$$

Using these expressions, the area of the segment can be written as: $\cos^{-1}\left(\frac{r-h}{r}\right) r^2 - (r-h)\sqrt{h(2r-h)}$

Because the can is perfectly cylindrical, the proportion of the cross-sectional area taken up by the liquid is also the proportion of the volume of the can that is taken up by the liquid. Finding this proportion answers the question 'How full is the can?'. The following expression tells us how full the can is:

$$\frac{\cos^{-1}\left(\frac{r-h}{r}\right) r^2 - (r-h)\sqrt{h(2r-h)}}{\pi r^2}$$

Puzzle 2

It takes an infinite amount of time for the ball to have a speed of zero, and stop.

How far does it travel before it stops?

The distance travelled by the ball is the area under the speed-time graph for the motion of the ball. Assuming the ball halves its speed uniformly over each set of 5 seconds, then we can find the area under the resulting graph as a series of trapeziums, with a trapezium for every 5 second interval.

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Puzzle 2 (continued)

There are an infinite number of trapeziums whose areas we need to find: the total area of the trapeziums can be simplified to the following infinite sum:

$$\text{area} = \frac{5}{2}\left(1 + 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)\right)$$

If you know about infinite geometric series you can use the formula below to find the value of the infinite sum in the inner bracket:

$$S = \frac{a}{1 - r}$$

In this formula, a is the first term and r is the multiplicative factor between each successive term.

In this case both a and r = $\frac{1}{2}$ and therefore S = 1. Replacing the sum in the inner bracket with the value 1 we find the area under the graph (and therefore the total distance travelled) is 7.5m. It takes an infinite amount of time to stop, but travels a finite distance!

Puzzle 3

Put 10 coins in one group and 16 coins in the other, and then turn all of the 10 over.

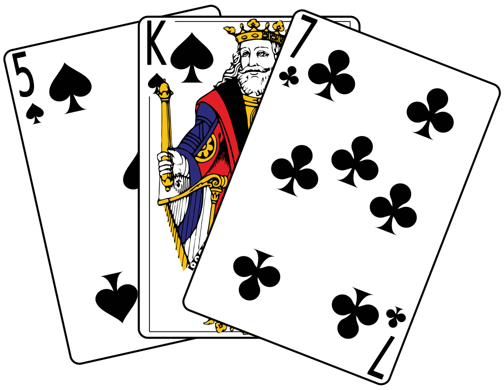
Let's imagine that to start with, **y** of the group of 10 are heads up. Then that means **10-y** of that group are tails up. When you turn them over, these results are reversed and in that group **y** coins are now tails up and **10-y** are heads up.

Now consider the group of 16. If **y** of the 10 heads up coins were originally put into the first group (of 10), then the remaining **10-y** heads up coins must be in the second group (of 16). And therefore after the group of 10 has been turned over, both groups have **10-y** heads up coins.

Puzzle 4

There must be (at least one) more of either red or black in your hand, as the number of reds plus number of blacks is one higher in your hand than in Rob's.

There is no reason that red or black would be more likely to be the one you have more of, so the chance you have more black is equal to the chance you have more red. Both have a 50% chance of happening.



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Puzzle 5

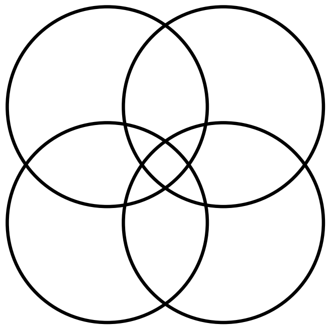
As all the spiders are saying different numbers, at most one of them can be telling the truth.

If they were all lying, there would be 4 spiders with 7 legs each, which gives a total of 28. But that leads us to a contradiction because one of the answers is 28, so one of the spiders would in fact be telling the truth.

Because they cannot all be lying - there must be exactly one of them telling the truth. This means 3 of them are lying giving 21 legs between those 3 spiders. The truth telling spider must have either 6 or 8 legs, bringing the leg total to either 27 or 29.

The truth must be amongst the spiders' answers because one of them is telling the truth; therefore the spiders have a total of 27 legs, and the 3rd spider (who has 6 legs) is telling the truth.

Puzzle 6



There are 14 regions. But if the numerical pattern continued, there would be 16 regions. You could justify this pattern by arguing that in theory there would be: 1 (4 choose 0) outside regions + 4 (4 choose 1) single regions + 6 (4 choose 2) double regions + 4 (4 choose 3) triple regions and 1 (4 choose 4) with all four circles.

However, in practice, it's physically impossible to draw the four circles so that circles that are 'diagonal' to each other intersect just with each other, and not also with a third circle. This reduces the number of regions by two.

Puzzle 7

Call the first and last digit a, and the middle digits b. You have 1000a + 100b + 10b + a
= 1001a + 110b
= 11(91a + 10b)
The number is therefore always divisible by 11.

Puzzle 8

The four possible sums each exclude one of the four digits (a,b,c and d), so each of them is involved in three of the four sums. Therefore: 14 + 16 + 19 + 20 = 3a + 3b + 3c + 3d
69 = 3(a + b + c + d)
23 = a + b + c + d

You can then work out the difference between each sum of three and the total of all four to find out what must have been missing. For example, the sum 20 must have been made without the number 3, and therefore one of the numbers is 3. The numbers on the cards are 3,4,7 and 9.