# PI AS A CONTINUED FRACTION

We celebrate $\pi$ approximation day on 22 July (22/7) because $\frac{22}{7}$ is an approximation to $\pi$.

$$\frac{22}{7} \approx 3.14286$$

This approximation can be found by following the method below.

**Step 1:** Re-write the whole number and fractional parts of $\pi$ separately

$$\pi \approx 3.14159$$

$$\pi \approx 3 + 0.14159$$

$$\boldsymbol{\pi \approx 3 + \frac{1}{n_0}}$$

**Step 2:** Find $n_0$

$$\frac{1}{n_0} \approx 0.14159$$

$$0.14159 n_0 \approx 1$$

$$n_0 \approx \frac{1}{0.14159}$$

$$n_0 \approx 7.062646$$

**Step 3:** Take the whole number part as $n_0$ and use in formula for $\pi$

$$n_0 \approx 7$$

$$\pi \approx 3 + \frac{1}{7}$$

**Step 4:** Write as one fraction

$$\pi \approx \frac{22}{7}$$

But we can improve this estimate!

In step 3, instead of taking just the whole number 7 as $n_0$ we could find a better approximation to $n_0$, by using the same process of approximation as described above. See below.

**Step 1:** Re-write the whole number and fractional parts of $n_0$ separately

$$n_0 \approx 7.062646$$

$$n_0 \approx 7 + 0.062646$$

$$\boldsymbol{n_0 \approx 7 + \frac{1}{n_1}}$$

**Step 2:** Find $n_1$

$$\frac{1}{n_1} \approx 0.062646$$

$$0.062646n_1 \approx 1$$

$$n_1 \approx \frac{1}{0.062646}$$

$$n_1 \approx 15.962711$$

**Step 3:** Take the whole number part as $n_1$ and use in formulae for $n_0$ and $\pi$

$$n_1 = 15$$

$$n_0 \approx 7 + \frac{1}{15}$$

$$\boldsymbol{\pi \approx 3 + \frac{1}{7 + \frac{1}{15}}}$$

**Step 4:** Write as one fraction

$$\pi \approx 3 + \frac{1}{\frac{106}{15}}$$

$$\pi \approx 3 + \frac{15}{106}$$

$$\pi \approx \frac{333}{106}$$

But we can still improve this estimate!

Again, instead of taking the whole number 15 as the value of $n_1$ we could use the approximation method to find a better approximation for $n_1$. If we did this, our estimate for $\pi$ would look as below.

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}}$$

This process of improving our estimate can continue ad infinitum, each time giving us a continued fraction with one more level of detail.

We can write the continued fraction above as:

$$\pi = [3; 7,15,1]$$

The first five rational approximations to $\boldsymbol{\pi}$ are listed below, in various forms.

| **Continued fraction** | **Continued fraction as a list** | **One fraction** | **Decimal expansion** | **Decimal places of accuracy** |
|---|---|---|---|---|
| $3 + \frac{1}{7}$ | $[3; 7]$ | $\frac{22}{7}$ | 3.14285714285 | 2 |
| $3 + \cfrac{1}{7 + \cfrac{1}{15}}$ | $[3; 7,15]$ | $\frac{333}{106}$ | 3.14150943396 | 3 |
| $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$ | $[3; 7,15,1]$ | $\frac{355}{113}$ | 3.14159292035 | 6 |
| $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$ | $[3; 7,15,1,292]$ | $\frac{103{,}993}{33{,}102}$ | 3.14159265301 | 8 |
| $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1}}}}}$ | $[3; 7,15,1,292,1]$ | $\frac{104{,}348}{33{,}215}$ | 3.14159265392 | 9 |