# ORBITAL MECHANICS

Mathematics describes how the planets stay in orbit around the Sun and enables us to send astronauts up in to space and to keep our own man-made objects in orbit. Here we have a look at that maths!

**How do the planets stay in orbit?**

To answer this we'll need two famous equations.

*Newton's Second Law of Motion:*

$$F = ma$$

where $F$ is the force acting on a body, $m$ is the mass of that body and $a$ is the acceleration of that body. This law tells us a body will not accelerate (start moving, or change speed or direction) unless a force is applied to it.

*Newton's Law of Universal Gravitation:*

$$F = \frac{GMm}{r^2}$$

where $F$ is the force between two point masses, $G$ is the gravitational constant, $M$ is the mass of the first body, $m$ is the mass of a second and $r$ is the distance between the centres of the masses. This law tells us that any two masses exert a force on each other (gravitational force).

When applied to the astronomical world these two equations explain how the planets remain in orbit; the force between any planet and the Sun (as dictated by the Law of Universal Gravitation) causes the planet to have an acceleration (by the Second Law of Motion).

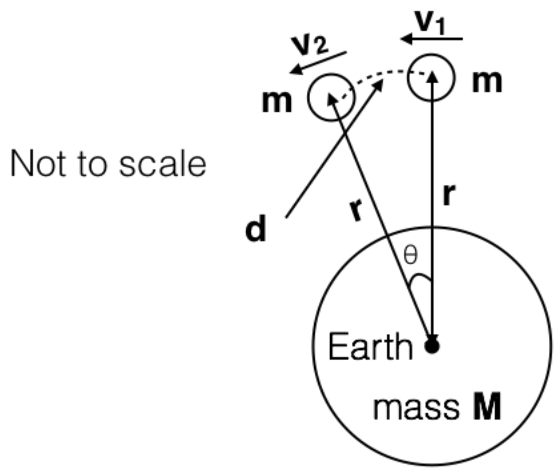
Since the mass of the planet is small (relative to the massive Sun), its acceleration due to the gravitational force is huge (the Second Law of Motion tells us that mass and acceleration are inversely proportional for a given force) and this large acceleration manifests as the planet continually changing its direction to travel around the Sun in an elliptical orbit.

And things that we send up into space stay in orbit around Earth for the same reasons!

**Orbital Mechanics for Space Missions**

A closer look at those two equations and the maths of things in orbit reveals a relationship between the distance a spacecraft is from the planet it is orbiting, and the speed it is travelling with.

This relationship is used by space scientists to keep spacecraft in orbit of Earth and to land them on other orbiting bodies, such as the moon and space stations.




The diagram above shows a spacecraft in orbit around the Earth. It is shown at two different positions in its orbit.

The Earth has mass $M$, the spacecraft has mass $m$ and velocity $v_1$ at the first position and velocity $v_2$ at a later position, and the distance between the Earth and the spacecraft is $r$ (which is constant at all points in the orbit).

The speed of a spacecraft in orbit does not change - the magnitudes of the two velocities are the same, and that is the speed of the spacecraft, $v$.

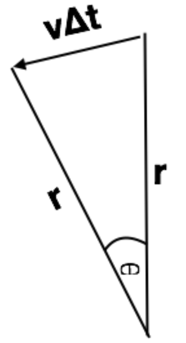
$$|v_1| = |v_2| = v$$

The arc distance ($d$) travelled by the spacecraft can be expressed as:

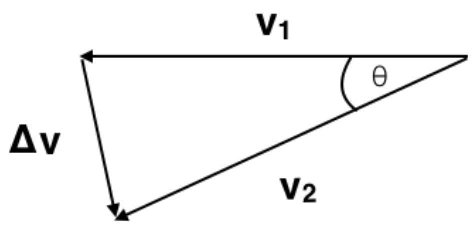
$$d = v\Delta t$$

When the angle $\theta$ is very small and the change of the position of the spacecraft is minimal, we can model the arc length as a straight line.

The triangle below shows the relationship between $r$,$\theta$ and $v\Delta t$.




And this triangle shows the two velocity vectors and the change in velocity. (The change in velocity can be found by $v_2 - v_1$).




The triangles are similar; both triangles are isosceles and they share the angle $\theta$ between those equal length sides.

Since the triangles are similar this means the ratio between the sides is the same in each triangle and we can write:

$$\frac{\Delta v}{v\Delta t} = \frac{v}{r}$$

Multiplying both sides by $v$:

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

Acceleration is the rate of change of velocity so can be expressed as:

$$a = \frac{\Delta v}{\Delta t}$$

$$a = \frac{v^2}{r}$$

By revisiting Newton's Second Law of Motion:

$$F = ma$$

$$F = m\frac{v^2}{r}$$

The force on the spacecraft is the gravitational force between the Earth and the spacecraft:

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

Simplifying (dividing both sides by $m$, and multiplying both sides by $r$) gives:

$$\frac{GM}{r} = v^2$$

This can be re-arranged to:

$$v^2 = GM\left(\frac{1}{r}\right)$$

G and M are both constants (the gravitational constant and the mass of the Earth respectively).

The relationship tells us that $v^2$ is inversely proportional to $r$, which means that the higher the spacecraft's orbit is, the smaller its speed will be, and visa versa.

Without this observation we would not be able to keep spacecrafts in orbit or perform landing operations.

Find out more about how this maths is used in space in Matt Parker's interview at NASA with astronaut Chris Hadfield, here: https://youtu.be/PooFvQEN4n8