THE ROCKET EQUATION

A rocket requires a great amount of fuel to power its journey into space. However, of course that fuel has mass, and so more fuel is then needed to provide power to overcome the mass of the original fuel, and so on. A fine balance is needed between the mass of the rocket and the mass of the fuel.

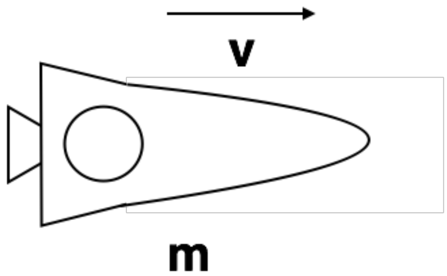
The 'Rocket Equation' allows us to find that balance and to calculate how much fuel is needed to get a certain mass rocket into orbit.

A rocket's journey into space is a continuous process of it ejecting fuel to provide it with power. We can derive the 'Rocket Equation' by considering that process and momentum before and after an instance of fuel ejection.

Deriving the Rocket Equation

Before fuel ejection

Not to scale

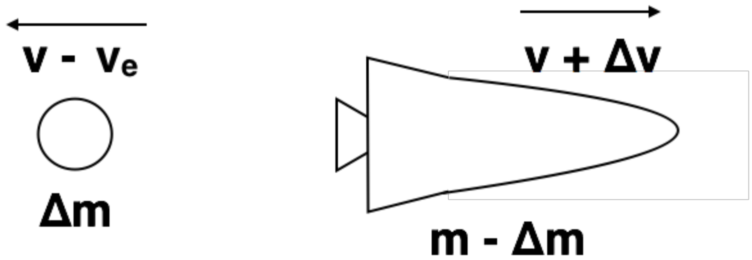


Above is a rocket with some small amount of fuel onboard (represented by the circle). The rocket (with the fuel) has mass m and velocity v.

Momentum can be found by multiplying mass by velocity.

$$momentum\ of\ rocket\ with\ fuel = mv$$

After fuel ejection



Now this image above shows the situation once the rocket has ejected the fuel. The fuel has mass Δm and the rocket has mass $m - \Delta m$.

The velocity of the fuel is $v - v_e$ (that is, the speed it was originally moving at, minus the speed it was ejected with). We can assume v_e is constant regardless of the mass of the fuel being ejected or the position in the rocket's journey.

And the velocity of the rocket has increased to $v + \Delta v$.

The momentum is the sum of the momentum of the fuel and the momentum of the rocket.

$$momentum\ after\ fuel\ ejection = (m - \Delta m)(v + \Delta v) + \Delta m(v - v_e)$$

The law of the conservation of momentum tells us that the momentum of a system remains constant, therefore:

$$mv = (m - \Delta m)(v + \Delta v) + \Delta m(v - v_e)$$

Expanding the brackets gives:

$$mv = mv - \Delta mv + m\Delta v - \Delta m\Delta v + \Delta mv - \Delta mv_e$$

This simplifies to:

$$0 = m\Delta v - \Delta m\Delta v - \Delta mv_e$$

$\Delta m\Delta v$ is a small change in mass multiplied by a small change in velocity, which will give a result so tiny that we can ignore this term without losing much accuracy.

$$m\Delta v = \Delta mv_e$$

We can re-arrange as follows:

$$\Delta v = v_e \frac{1}{m} \Delta m$$

This relationship only applies over a small change in mass (Δm) of the rocket. We're interested in finding a relationship that applies for the whole journey.

To do this we can integrate both sides of the equation over the whole journey of the rocket from launch to when it is finally in orbit. This means integrating between intial velocity and final velocity and between initial mass and final mass.

By convention both dm and dv should signify increases in the mass and velocity, and therefore we need to change the sign of dm below.

$$\int_{v_0=0}^{v_f} dv = -v_e \int_{m_0}^{m_f} \frac{1}{m}\, dm$$

$$_{v_0=0}^{v_f}[v] = -v_e\ _{m_0}^{m_f}[\ln{(m)}]$$

$$_{v_0=0}^{v_f}[v] = -v_e\ _{m_0}^{m_f}[\ln{(m)}]$$

$$v_f = -v_e\left(\ln\left(m_f\right) - \ln{(m_0)}\right)$$

By using log laws:

$$v_f = -v_e \ln\left(\frac{m_f}{m_0}\right)$$

And raising both sides as powers to the base e:

$$\frac{m_f}{m_0} = e^{-\frac{v_f}{v_e}}$$

This is the ‘Rocket Equation’!

Using the Rocket Equation

As we can see above the Rocket Equation gives us a relationship between the mass of the rocket when it is finally in orbit without all its fuel (m_f), the mass of the rocket initially with all its fuel on board (m_0) and the final velocity (v_f) the rocket needs to achieve in order to stay in orbit.

Any shuttles trying to dock on to the International Space Station would need to be orbiting around the Earth at the same speed as the ISS which is around $v_f = 8000ms^{-1}$, and we can assume the speed the fuel is ejected with is $v_e = 3500ms^{-1}$.

We can substitute these two values into the equation above:

$$\frac{m_f}{m_0} = e^{-\frac{8000}{3500}}$$

$$\frac{m_f}{m_0} \approx 0.102$$

We've found that the ratio between the mass of the rocket with its fuel and the mass of the rocket without fuel is around 0.102

This tells us how much fuel we need to get a rocket of any mass into orbit around the Earth!

Bonus activity

The Saturn V shuttle had a payload mass (mass without fuel) of 140,000kg.

Can you use the formula above to work out what mass of fuel was needed to take it into orbit at $8000ms^{-1}$?

The velocity needed to escape the gravitational pull of the Earth (escape velocity) is about $11{,}190ms^{-1}$ - far greater than the velocity needed to orbit the Earth.

Can you calculate how much extra fuel is required to get the shuttle (the same payload) to escape velocity?