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THE ONE HUNDRED DRAWERS

A cabinet with a hundred drawers, each numbered from 1 to 100, stands in an otherwise empty room in a prison. The warden enters the room, writes down the names of 100 prisoners on separate pieces of paper, and distributes the pieces of paper randomly in the cabinet so that each drawer contains the name of one prisoner.

The warden exits the room and assembles the 100 prisoners whose names she wrote on the slips of paper. She tells them that they will each be let into the cabinet room, one after the other; once in there they are permitted to open 50 drawers and look at the names on the pieces of paper inside. She adds that if every single prisoner opens the drawer containing his name, everyone will be freed. But if at least one prisoner doesn't open the drawer with his name in it, everyone will be killed.

The prisoners are allowed to think up a strategy before they go into the cabinet room, but once the first prisoner goes in they are not allowed to communicate in any way. They cannot leave messages in the room, nor tell prisoners what they saw once they leave the room.

Can you find the strategy that gives a chance of survival of more than 30 per cent?

Of all the mathematical surprises in this book, this result is perhaps the most staggering. If a prisoner chooses 50 out of 100 boxes at random, the chance that one of these boxes contains his name is 50 out of 100, or 50 per

cent. If all 100 prisoners were to choose 50 boxes at random, the probability of everyone choosing their name would be:

50 per cent of 50 per cent of 50 per cent … (repeated 100 times) … which is:

0.00000000000000000000000000008 per cent

Yet a strategy exists that improves the chances of *every* prisoner finding his name by a factor of more than 100 octillion.

The strategy is very simple to describe, and relies on a very interesting piece of mathematics that is fully explained in the back of the book. Even if you can't prove that it gives the prisoners more than a 30 per cent chance of survival, have a guess as to what it might be. You will be genuinely astounded at how a collective decision can beat the odds.

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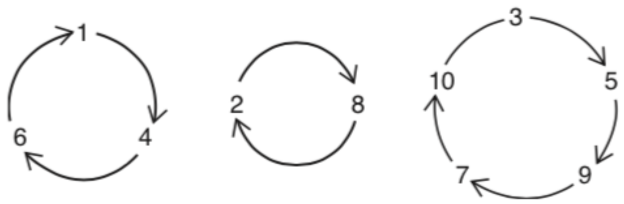
Before we get to the solution, here's an introduction to the maths of permutations. It's going to make the prisoners' strategy much easier to understand. Let's say we have 10 objects and we want to reorder them. One way to describe this reordering is in a table:

Initial position	1	2	3	4	5	6	7	8	9	10
New position	4	8	5	6	9	1	10	2	7	3

An easy way to understand the pattern described in the table is to represent it visually. In the table 1 → 4, 4 → 6 and 6 → 1. This forms a loop, or 'permutation cycle'. The table can be illustrated thus:



It's plain to see that there are three cycles, one of length 3, one of length 2, and one of length 5. There are more than 3.6 million permutations of 10 objects, and they can contain cycles of length 1 to length 10.

Now back to the prisoners. The strategy that they must take is the following. First, they must number themselves from 1 to 100 – that is, they should order themselves into Prisoner 1, Prisoner 2, Prisoner 3, and so on.

They then must agree to abide by these rules when they enter the room:

[1] Each prisoner heads for the drawer with their number on it and opens it first. In other words, the first drawer that Prisoner 1 opens is drawer 1, the first drawer that Prisoner 2 opens is drawer 2, and so on.

[2] If a prisoner opens a drawer and it contains the name of another prisoner, say Prisoner *k*, the next drawer they open should be drawer *k*. In other words, if Prisoner 1 opens a drawer with Prisoner 32's name in it, the next drawer he should open is drawer 32. If the drawer has Prisoner 67's name in it, the next drawer he should open is drawer 67, and so on.

These two rules set each prisoner on a path that is equivalent to a permutation cycle.

Here's why. Let's imagine there are only 10 drawers and 10 prisoners. If the top row of the table on the left is relabelled 'drawer number' and the bottom row is relabelled 'prisoner number', the table now describes one possible arrangement of the prisoners' names in the drawers. So, for example, in drawer 1 is the name of Prisoner 4, in drawer 2 is the name of Prisoner 8, and so on.

Drawer number	1	2	3	4	5	6	7	8	9	10
Prisoner number	4	8	5	6	9	1	10	2	7	3

If Prisoner 1 abides by the rules in this strategy, he begins by opening drawer 1, in which he finds the name of Prisoner 4. So he opens drawer 4, in which he finds the name of Prisoner 6, so he opens drawer 6, in which he finds his own name. He has gone through the cycle 1 → 4 → 6 → 1 , and has found his own name after opening three drawers.

We can see what happens to the other prisoners by following the cycles of this permutation (illustrated on the previous page.) Prisoners 4 and 6 will also find their names after three drawers, Prisoners 2 and 8 will find their names after two, and the others will find their names after five. In other words, the number of drawers a prisoner must open to find his own name is equal to the length of the permutation cycle he finds himself in. The strategy counts him round the cycle drawer by drawer, and he finds his name when he completes the cycle.

This observation is also true when we increase the number of drawers and prisoners to 100.

If there are 100 prisoners, and each prisoner can only open 50 drawers, every prisoner will find his own name if and only if all the permutation cycles have a length of at most 50. If the length of a cycle is longer than 50, a prisoner won't be able to travel round it in only 50 drawer-openings.

The strategy works, therefore, if no permutation cycles are longer than 50. In other words, in order to work out the chances of freedom for all the prisoners – that is, of everyone finding their names – we need to calculate the chances of there being no permutation cycles longer than 50 in a random permutation of 100 objects. (In other words, we need to divide the number of permutations of 100 objects that have no permutation cycles longer than 50 by the total number of permutations of 100 objects.) The maths here is too technical for a book like this one, so you'll need to trust me that the chances of there being no permutation cycles longer than 50 are just over 30 per cent.

The interesting behaviour of permutation cycles gives the prisoners an unexpectedly good chance of survival.