# TEACHERS NOTES AND SOLUTIONS

Although all the activities in this set are linked in some way and complement each other, some tasks follow on from each other more strictly than others. If you plan to do only a subset of the activities, bear in mind which activities are prerequisites for others.

The activities are in order in the list below, with any prerequisite activities for each listed next to it in brackets.

- Graphs and shapes
- Euler Characteristic Formula (*Graphs and shapes*)
- Graphs and patterns (*Graphs and shapes*)
- Famous graph puzzles
- Mug solutions (*Famous graph puzzles*)
- Utilities Puzzle proof (*Graphs and shapes, Euler Characteristic Formula and Famous graph puzzles*)

The notes on the next pages feature solutions, hints and background information for each of the worksheets. The notes for each worksheet are started on a new page.

# GRAPHS AND SHAPES

All convex polyhedra can be represented as planar graphs! A face needs to be chosen that will effectively stretch around the others to make it the outside edges of the graph, whilst the other faces are represented as regions within that. Students could investigate other graphs made from convex polyhedra.

**Question 1 and 2**

The table below gives the answers to both the match up exercise and table exercise.

| Graph | Name of shape | V | F | E |
|---|---|---|---|---|
| **A** | Square based pyramid | 5 | 5 | 8 |
| **B** | Tetrahedron | 4 | 4 | 6 |
| **C** | Triangular Prism | 6 | 5 | 9 |
| **D** | Octahedron | 6 | 8 | 12 |

**Question 3**

The formula that connects V, E and F is $V + F – E = 2$. This is called the Euler Characteristic Formula. We've deliberately left space for students to add an extra column to the table. It might help them spot the pattern if they are directed to record $V + F$ in this extra column, and then compare it to their values for E.

# EULER CHARACTERISTIC FORMULA

**Question 1**

a) A graph with only two vertices needs only one edge and has only one face. These numbers satisfy the Euler Characteristic Formula.
b) Students should notice that they must add an edge when they add another vertex.
c) In the final diagram students should notice that when they add another edge between two vertices this creates another face.

**Question 2**

The only things that can be actively added when building a graph are a vertex or an edge. The observations above show that when either of these things happen the total of $V + F$ will increase by one and the value of $E$ will also increase by one. This means the value of $V + F – E$ will remain constant as vertices and edges are added. Since the simplest graph (with two vertices) gave $V + F – E = 2$ this means all other graphs will satisfy this formula too.

# GRAPHS AND PATTERNS

The graphs on this worksheet **do not need to be planar** - the edges are allowed to cross over.

**Question 1 - Complete graphs**

a) The graphs need 6 and 10 edges respectively.

b) $Number\ of\ edges = \frac{1}{2}m(m-1)$

This comes from the sum of all positive integers lower than or equal to $(m-1)$. The first vertex needs to join to $(m-1)$ vertices so you draw $(m-1)$ edges. Then the next vertex is already joined to the first one, so only another $(m-2)$ edges must be drawn to the second vertex, and a further $(m-3)$ edges to the third vertex, and so on.

The total number of edges is $(m-1)+(m-2) \dots + 1$ and this sum is equal to $\frac{1}{2}m(m-1)$. Numbers of this form are triangular!

**Questions 2 - Complete Bipartite graphs**

a) The graphs have 4, 9 and 12 edges respectively.

b) $Number\ of\ edges = nm$

**Questions 3 - Extension: Bipartite graphs**

Number of possible graphs = $2^{mn}$

Solution 1

It is not required that the graph is complete bipartite (although it could be), it is just required that it is bipartite. So, for each possible pairing, there are two options for the state they can be in - they could be connected or not connected.

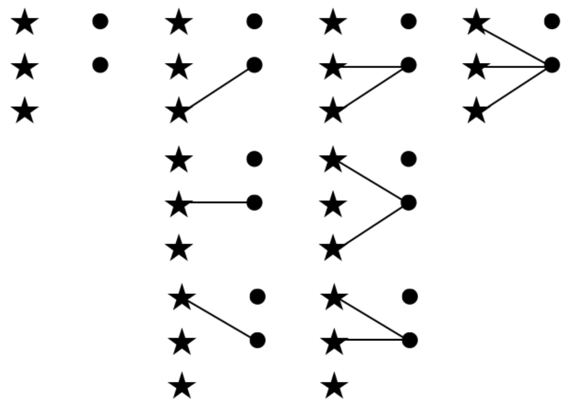
For every one of the n vertices in one set, there are m possible vertices in the other set that it can pair up with, so there are n x m possible pairings.

With two options for each of n x m pairings that gives us $2^{mn}$ possible graphs.

Solution 2

There is a second solution we have found that uses combinatorics and is a little more complex. It could be that A-level students try to approach the problem from this direction.

In the example below where m=3 and n=2, the diagrams show the possible edges that could be connecting the bottom dot vertex.



The number of ways of connecting the dot to two stars is the same as the number of ways of choosing two stars from three (3C2). Similarly, there are 3C0 ways of connecting the dot with no stars, 3C1 ways of connecting the dot with one star and 3C3 ways of connecting it with all three stars.

So, the total number of different ways the bottom dot vertex can be connected is: 3C0 + 3C1 + 3C2 + 3C3. This sum is equal to $2^3$.

If there were m stars rather than 3, this sum would be $2^m$. And the number of options for connecting any other dot vertex would be exactly the same: $2^m$.

So, the total number of ways the graph could be drawn is $(2^m)^n = 2^{mn}$ where n is the number of dot vertices.

We would get the same result if we considered how many ways to connect the star vertices first instead. We would end up with $(2^n)^m$ which also gives $2^{mn}$.

# FAMOUS GRAPH PUZZLES

This task can be done as a stand-alone task however if your students have looked at 'Euler Characteristic Formula' they will have met the idea of what a graph is and what it means for a graph to be 'planar'.

Also, if your students have done the 'Graphs and patterns' task they should recognise that the graph in the Utilities Puzzle is complete bipartite and that the graph $K_5$ is complete.

Page 2 and 3 of this file are a bigger version of each of the puzzles, in case you wish to laminate the diagram so students can draw and erase the graphs when doing the investigation.

If students are struggling to come up with ideas about which constraints to change, some ideas are below. You could ask the students:

- Can you solve the problem so the edges only cross in one place? (If you intend to go on to try to solve the puzzles on a mug then it's useful for students to have got to this point anyway).
- Can you solve the problem if you remove some of the vertices? What is the least number of vertices you can remove?

It is possible to find solutions to both puzzles if we change the constraints as described above.

However, the most fun constraint to change is that the puzzles must be solved on a flat surface. Sometimes people find solutions that involve folding the paper over. This should be encouraged! It makes the puzzle possible and just involves changing the surface that it's being solved on.

To get an idea for how changing the surface might help you solve the problem, have a look at 'Mug solutions'. Both puzzles can be solved on the surface of a mug!

# MUG SOLUTIONS

Both puzzles from ‘Famous graph puzzles’ can be solved on the surface of a torus, and therefore they can both be solved on the surface of a mug, because a mug is a like a deformed torus. The important (topological) aspect that is the same between the two shapes is that they both have one hole – in the case of the torus that’s the hole in the middle, and in the case of the mug it’s the hole in the handle.

Have a go using the ‘almost-solutions’ on the worksheets to solve the puzzles on a mug!

If you would like your class to have a go at the Utilities Puzzle on a mug then, should they all bring in a mug they are allowed to draw on (!) and have access to dry-wipe pens, you can print out our ‘Utilities Puzzle Stickers’ sheet so they can stick the symbols on to their mug. Students get a label each and will need cut the vertices out separately and stick them on the mug.

The sticker sheet is designed to fit on to the following Avery stickers:
**Avery L7160-40, 63.5 x 38.1 mm labels, 21 per sheet.**

They can be purchased here:
https://www.labelplanet.co.uk/proddetail.php?prod=LP21/63 and from Avery here: https://www.avery.co.uk/product/address-labels-l7160-40

Bingo! Now everyone can solve the Utilities Puzzle on their mug!

Alternatively, they could of course just draw their own versions of the house and utilities symbols on their mug.

Also, Utilities Puzzle mugs complete with dry wipe pen can be purchased on Maths Gear here: https://mathsgear.co.uk/products/utilities-puzzle-mug

# UTILITIES PUZZLE PROOF

Steps of the proof, with the gaps filled in:

- Number of vertices = 6, Number of edges = 9 (3 for each of the 3 houses).
- Therefore, if V + F – E = 2, this implies V + F = 11, and F = 5.
- Each face is bordered by at least 4 edges.
- Every edge is shared by 2 faces.
- This means there are at least 2 edges per face, so with 5 faces, that's at least 10 edges in total, $E \geq 10$
- This contradicts our initial observation that there must be 9 edges.

This video on YouTube https://www.youtube.com/watch?v=VvCytJvd4H0 is worth a watch for a slightly different proof that you can show your students (one that doesn't involve the Euler Characteristic Formula).

A similar proof to our one above can be constructed to **prove that $K_5$ is non-planar.**

- Number of vertices = 5, Number of edges = 4 + 3 + 2 + 1 = 10 (links to 'Graphs and patterns').
- Therefore, if V + F – E = 2, this implies V + F = 12, and F = 7.
- Each face is bordered by at least 3 edges.
- Every edge is shared by 2 faces.
- This means there are at least 1.5 edges per face, so with 7 faces, that's at least 10.5 edges in total, $E \geq 10.5$
- This contradicts our initial observation that there must be 10 edges, which means that V + F – E = 2 must not apply, and our graph is non-planar.

Other proofs for the non-planarity of both graphs do exist! It's also interesting to note that all non-planar graphs contain at least one of these two famous graphs!