Psychic Pets Teacher Notes and Solutions

Worksheet 1: Psychic Pets Predict World Cup Winners

This worksheet is aimed at students from within Key Stage 3, Key Stage 4 or Key Stage 5. Students will also need the World Cup Fact Sheet.

1) Write out the games for teams A,B,C and D. The games are A v B, A v C, A v D, B v C, B v D and C v D. There are 3 + 2 + 1 = 6 games (Or alternatively, 4C2 = 6).

2) Start by writing out how many possible outcomes there are for just two games. For example, AD is one outcome – it represents 'Team A win game 1' and 'it's a draw in game 2'.
There are $3^6 = 729$ outcomes for 6 games.

3) 6 x 8 = 48

4) 3 options per game. $3^{48} = 7.98 \times 10^{22}$

5) Use the World Cup Factsheet to count there are 16 knock out games (don't forget the third place play-off). Each with 2 options. 2^{16} = 65536

The number of outcomes over the whole tournament is the number of outcomes at group stage multiplied by the number of outcomes in the knock-out stage: 7.98 x 10^{22} x 65536 = 5.23 x 10^{27}.
That's more than one hundred quadrillion (10^{17}) pets for every person in the world!

6) Before your students start you may wish to break the question down into sub-questions like this:

How many outcomes are there for England's 3 group stage games?
There are 3 options for each of 3 games, so there are 3^3 = 27 outcomes at the group stage.

How many outcomes are there for England's knock-out stage games?
There are 4 rounds with two options for each round, but there are not 2^4 outcomes in the knock-out stage. If, for example, England lose a quarter-final, they do not go on to play a semi-final. So some of the 2^4 outcomes are impossible. For example WLWW is impossible - as no games would be played after the quarter final loss. However, semi-final loss does still result in a game – the third place play-off.

Taking this into account the number of outcomes can be counted as the number of exit points during the knock-out stage. England can exit in 6 ways if they enter the knock-out stage: after losing the first round of the knock-out stage, after losing the quarter final, after losing the third place play-off, after winning the third place play-off, after losing the final or after winning the final.

You could leave it at this, and multiply 27 (group game outcomes) by 6 (knock-out outcomes). However…

Do all group stage outcomes result in England entering the knock-out stage?
No! For example, if England lose all 3 group stage games they will not progress to the knock-out stage and therefore LLLL is an impossible outcome. If you wish to account for this, give your students the following information:

- 4 of the 27 group stage outcomes result in England definitely going through (WWW and all the permutations of WWD)
- 19 of the 27 group stage outcomes result in England maybe going through (all the permutations of WWL, WDD, WDL, WLL, DDD or LDD)
- 4 of the 27 group stage outcomes (LLL and all the permutations of LLD) result in England definitely being knocked out.

Can students work out how to adjust their calculations with this new information?

Our solution - we've categorised all England's possible 'World Cup journeys' in this way:

Possible journeys where they are knocked out in group stage: 4 + 19 = 23
(Eg one of the 19 journeys used here could be described as LDW and exit at group stage)
Possible journeys where they progress to knock-out stage: (19 + 4) x 6 = 138
(There are 19 + 4 ways they could progress to the knock-out stage, and then any of the 6 knock-out stage outcomes can occur).
Total number of journeys: 23 + 138 = 161

However, we would only need 161 – 19 = 142 pets at most, for one of them to predict 'England's World Cup journey' correctly. This is because in real life pets are not predicting everything before the World Cup starts; their owners will wait to see if their team goes through to the knock-out stage before the pet makes a prediction. This means we only need to count the 19 'could go either way' options once.

Worksheet 2: How Many Psychic Pets Do We Need?

This worksheet is aimed at students from within Key Stage 4 and Key Stage 5. They will also need the Psychic Pets Results sheet. Our solutions for this worksheet involve combinatorics, but the questions can also be solved with the use of tree diagrams.

1) The total number of ways the 4 pets can make their choices is 4^4. There is $4C4 = 1$ way of them choosing one outcome each. And there are 4! ways of ordering these choices (it matters which pet chooses which outcome). So, the probability they cover all 4 different outcomes is $\frac{4!}{4^4} = 0.09375$

Extension: The probability that all 4 outcomes are covered for 5 pets:
Total number of ways 5 pets can choose is 4^5. For 5 pets to cover 4 outcomes, one must be chosen twice. There are 4C1 ways of choosing the repeated one. There are 5! ways of ordering 5 distinct choices, but one is repeated so we divide 5! by 2! (as this is the number of ways of ordering the 2 repeated ones). So, the probability 5 pets cover all 4 different outcomes is $\frac{4C1 \times 5!}{4^5 \times 2!} = 0.234375$

The probability that all 4 outcomes are covered for 6 pets:
Total number of ways 6 pets can choose is 4^6. For 6 pets to cover 4 outcomes, one outcome can be chosen 3 times, or 2 outcomes can be chosen twice each.

Number of ways one outcome is chosen 3 times: 4C1 ways of choosing the repeated one. There are 6! ways of ordering 6 distinct choices, but as 3 are identical we divide by 3! So, we use $\frac{4C1 \times 6!}{3!}$

Number of ways 2 outcomes are repeated: 4C2 ways of choosing the 2 repeated outcomes. To find the number of ways of ordering the 6 choices divide 6! by 2! x 2! to account for having two identical pairs. So, use $\frac{4C2 \times 6!}{2! \times 2!}$

Probability of 6 pets covering 4 outcomes = $\frac{1}{4^6}\left(\frac{4C1 \times 6!}{3!} + \frac{4C2 \times 6!}{2! \times 2!}\right) = 0.380859375$

2) Covering 3 outcomes: $4C3$ is the number of ways of choosing the 3 outcomes that get covered. In each case one outcome must be chosen twice. There are $3C1$ ways of choosing the repeated one. There are 4! ways of ordering 4 distinct choices, but one is repeated so we divide 4! by 2!

So, the probability the 4 pets cover 3 outcomes is $\frac{4C3 \times 3C1 \times 4!}{4^4 \times 2!} = 0.5625$

Covering 2 outcomes: $4C2$ is the number of ways of choosing the 2 outcomes that get covered. In each case one outcome can be chosen three times, or both outcomes can be chosen twice.

Number of ways one outcome is chosen 3 times: There are $2C1$ ways of choosing the repeated one. The number of ways of ordering the 4 choices is 4! divided by 3! to account for having one choice appearing 3 times.

So, we use $\frac{4C2 \times 2C1 \times 4!}{3!}$

Number of ways of both outcomes being chosen twice: The number of ways of ordering the 4 choices is 4! divided by 2! x 2! to account for two repeated pairs.

So, we use $\frac{4C2 \times 4!}{2! \times 2!}$

Probability the 4 pets cover 2 outcomes = $\frac{4C2}{4^4}\left(\frac{2C1 \times 4!}{3!} + \frac{4!}{2! \times 2!}\right) = 0.328125$

Covering 1 outcome: $4C1$ is the number of ways of choosing the 1 outcome that gets covered. Since that outcome is chosen by all 4 pets, there is only one way of ordering the 4 choices.

So, the probability the 4 pets cover only 1 outcome is $\frac{4C1}{4^4} = 0.015625$

3) P($correct\ pets \geq 1$) = 1 - P($correct\ pets = 0$)

If, for example, only 3 outcomes are covered the chance that no pets are correct is $\frac{1}{4}$. To find the chance no pets are correct we need to consider all the possible outcome coverage options, as below. Here X is the number of outcomes covered by the pets' choices.

P($correct\ pets = 0$) = (P($X = 4$)× 0 + P($X = 3$) ×$\frac{1}{4}$ + P($X = 2$) × $\frac{1}{2}$ + P($X = 1$)×$\frac{3}{4}$)

P($correct\ pets \geq 1$) = 1 - (P($X = 4$)× 0 + P($X = 3$) ×$\frac{1}{4}$ + P($X = 2$) × $\frac{1}{2}$ + P($X = 1$)×$\frac{3}{4}$)

Finding P($X = n$) values from question 2 (or from Psychic Pets Results sheet)

P($correct\ pets \geq 1$) = 1 – 0.31640625

= 0.68359375

4) Students should use the Psychic Pets Results Sheet to get the P(X=n) values for more than 4 pets. When there are 8 pets the probability of at least one correct pet goes above 90% for the first time with 90.025%.

In case anyone tries by hand, our calculations for the P(X=n) values for 5 pets are below.

$P(X=4) = \frac{4C1 \times 5!}{4^5 \times 2!}$

$P(X=3) = \frac{4C3}{4^5}\left(\frac{3C1 \times 5!}{3!} + \frac{3C2 \times 5!}{2! \times 2!}\right)$

$P(X=2) = \frac{4C2 \times 2C1}{4^5}\left(\frac{5!}{4!} + \frac{5!}{3! \times 2!}\right)$

$P(X=1) = \frac{4C1}{4^5}$